# Energy-Momentum Tensor and Particle Creation in the de Sitter Universe

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Particle creation in a conformally flat spacetime (e.g., FRW universe) requires a non-conformal field. The choice of state is crucial, as one may misunderstand the physics of particle creation by choosing a too restrictive vacuum for the quantum field. We exhibit a vacuum state in which the expectation values of the energy and pressure allow an intuitive physical interpretation. We apply this general result to the de Sitter universe.



## I. INTRODUCTION

We first consider a charged scalar field $\Phi$ in Minskowski spacetime. Suppose that the electric field is $\mathbf{E} = E\mathbf{z}$, and the vector potential is $\mathbf{A}(t) = -Et\mathbf{z}$. The wave equation for a charged field in a Minkowski universe is given by:

$$[(\partial_\mu - ieA_\mu)^2 + m^2]\Phi(t,\mathbf{x}) = 0\,. \tag{1.1}$$

We can find solutions of the form (Fourier mode decomposition)

$$\Phi(t,\mathbf{x}) = \frac{1}{V^{1/2}}\sum_{\mathbf{k}} a_{\mathbf{k}} f_{\mathbf{k}}(t) e^{i\mathbf{k}\cdot\mathbf{x}} + b^\dagger_{-\mathbf{k}} f^*_{-\mathbf{k}}(t) e^{-i\mathbf{k}\cdot\mathbf{x}}\,, \tag{1.2}$$

with mode functions that satisfy the following harmonic differential equation:

$$\ddot{f}_{\mathbf{k}}(t) + \omega^2_{\mathbf{k}}(t) f_{\mathbf{k}}(t) = 0 \quad \text{where} \quad \omega^2_{\mathbf{k}}(t) = (k_z + eEt)^2 + k_\perp^2 + m^2\,. \tag{1.3}$$

These mode functions are invariant under time reversal, $(t \to -t, k_z \to -k_z)$, but the expectation value of the electric current is odd under this exchange. This is easy to see; suppose we consider solutions such that $|f_{\mathbf{k}}(t)| = |f_{-\mathbf{k}}(-t)|$. The expectation value of the current is

$$\langle j_z \rangle = \frac{2e}{V}\sum_{\mathbf{k}} (k_z + eEt)|f_{\mathbf{k}}(t)|^2\,. \tag{1.4}$$

Therefore in the vacuum state defined by $a_{\mathbf{k}}|0\rangle = 0 = b_{\mathbf{k}}|0\rangle$, we have $\langle j_z\rangle = 0$. On the other hand, we know that there are solutions with adiabatic asymptotic behaviour, such that

$$\lim_{t\to\pm\infty} f_{\mathbf{k}(\pm)}(t) = \tilde{f}_{\mathbf{k}}(t) \quad \text{with} \quad \tilde{f}_{\mathbf{k}}(t) = [2\omega_{\mathbf{k}}(t)]^{-1/2}\exp[-i\int^t dt'\omega_{\mathbf{k}}(t')]\,. \tag{1.5}$$

These two families of solutions $\{f_{\mathbf{k}(-)}\}$ and $\{f_{\mathbf{k}(+)}\}$ are related by a Bogoliubov transformation and they represent two different vacua. If our initial vacuum state is $|0_{(-)}\rangle$ (adiabatic vacuum at early times), it is easy to show that in the remote future, when the natural choice for a set of adiabatic observers is $\{f_{\mathbf{k}(+)}\}$, these *inertial observers* would detect particle production given by

$$N_{\mathbf{k}(+)}|0_{(-)}\rangle = [a^\dagger_{\mathbf{k}(+)} a_{\mathbf{k}(+)} + b^\dagger_{-\mathbf{k}(+)} b_{-\mathbf{k}(+)}]|0_{(-)}\rangle \neq 0\,. \tag{1.6}$$

These observers will measure a nonvanishing $\langle j_z\rangle$. This is the Schwinger effect which would be completely missed in the $T-$invariant vacuum state [1].

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## II. SCALAR FIELD IN DE SITTER SPACETIME

We next consider a scalar field $\Phi$ in a de Sitter gravitational background. We use a coordinate system [2] in which the spatial sections have curvature $\kappa = +1$, and the scale factor is $a(t) = Z^{-1}\cosh(Zt)$ with $u = Zt$. The wave equation is

$$[-\Box + m^2 + \xi R]\ \Phi(t, \mathbf{x}) = 0\,. \tag{2.1}$$

In the de Sitter universe this equation is separable and the field $\Phi$ can be written in terms of creation and annihilation operators. The equation of motion for the mode functions is

$$\ddot{f}_k(t) + \Omega_k^2(t) f_k(t) = 0\,, \tag{2.2}$$

with

$$\Omega_k^2(t) = Z^2\left[\gamma^2 + \left(k+\frac{1}{2}\right)\left(k+\frac{3}{2}\right)\mathrm{sech}^2(Zt)\right] \quad \text{and} \quad \gamma^2 = \frac{m^2}{Z^2} + 12\left(\xi - \frac{1}{6}\right) - \frac{1}{4}\,. \tag{2.3}$$

In the remote past and future $\Omega_k$ tends to a constant value, and we can find exact mode functions such that at early and late times they tend to the adiabatic ones. These two families of solutions $\{f_{k(-)}\}$ and $\{f_{k(+)}\}$ are related by a Bogoliubov transformation

$$f_{k(-)}(t) = \hat{\alpha}_k f_{k(+)}(t) + \hat{\beta}_k f^*_{k(+)}(t)\,, \tag{2.4}$$

where $|\hat{\beta}_k|^2 = \mathrm{cosech}^2(\pi\gamma) \neq 0$. We note that $|\hat{\beta}_k|^2$ is time independent, as well as $k-$independent. This is the analog of the Schwinger effect mentioned in the previous section [3,4]. Notice that in the de Sitter invariant vacuum there would no particle creation. This state is the analog of the $T$-invariant vacuum of the previous section.

The application of these results to the early universe requires finite-time initial conditions. Suppose we take the solution $f_{k(-)}$ as our initial condition at time $t_0$. When the gravitational background is curved, the set of adiabatic observers $\{\tilde{f}_k\}$ with

$$\tilde{f}_k(t) = [2\Omega_k(t)]^{-1/2}\exp[-i\int^t dt'\Omega_k(t')]\,, \tag{2.5}$$

is the closest we can get to the concept of *inertial observers*. In order to obtain a *particle interpretation* we write

$$f_k(t) = \tilde{\alpha}_k(t)\tilde{f}_k(t) + \tilde{\beta}_k(t)\tilde{f}^*_k(t)\,. \tag{2.6}$$

The number of particles is

$$\tilde{N}_k(t) = \langle 0_{(-)}|\tilde{a}_k^\dagger(t)\tilde{a}_k(t)|0_{(-)}\rangle = |\tilde{\beta}_k(t)|^2\,, \tag{2.7}$$

which depends on the value of $\gamma$ implicitly. We plot the results for $\gamma = 1$ and different values of $k$.

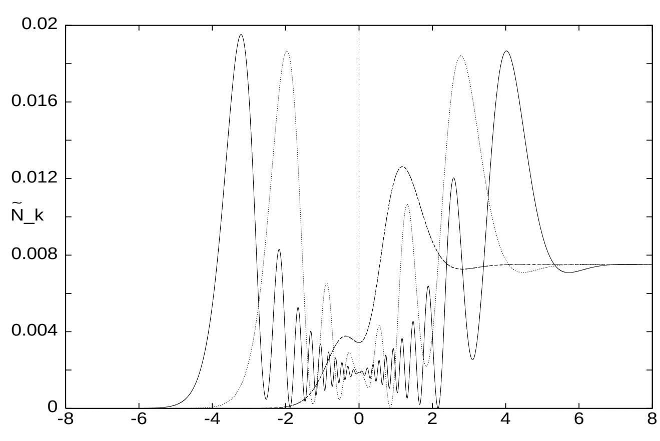


FIG. 1. Number of adiabatic particles for different values of $k$. The dashed, dotted, and solid lines are respectively $k = 0$, $k = 5$, and $k = 10$.

## III. ENERGY-MOMENTUM TENSOR

The most important physical quantity to compute is the expectation value of the energy-momentum tensor in the state defined by the modes $f_k$. We consider the case $\xi = 1/6$ and $m \neq 0$. It can be shown that there is a particular choice of adiabatic observers (those with zeroth adiabatic frequency) for which the energy takes the very simple form

$$\langle \epsilon \rangle_{\xi=1/6} = \frac{1}{(2\pi^2 a^3)} \sum_{k=0}^{+\infty} (k+1)^2 \frac{\omega_k}{2} \left(1 + 2N_k\right) . \tag{3.1}$$

The pressure can be obtained from the conservation equation

$$\dot{\langle \epsilon \rangle}_{\xi=1/6} + 3 \frac{\dot{a}(t)}{a(t)} \left[ \langle \epsilon \rangle_{\xi=1/6} + \langle p \rangle_{\xi=1/6} \right] = 0 . \tag{3.2}$$

The previous equations can be generalized for $\xi \neq 1/6$.

The energy-momentum tensor needs to be regularized and renormalized before obtaining its finite physical value. We regularize the energy-momentum tensor by adiabatic methods, since we know that the fourth order adiabatic energy and pressure have the same divergences as $\langle \epsilon \rangle$ and $\langle p \rangle$, respectively [5]. We define [6]

$$\langle \epsilon \rangle_R = \langle \epsilon \rangle_B - \langle \epsilon \rangle_A , \quad \langle p \rangle_R = \langle p \rangle_B - \langle p \rangle_A . \tag{3.3}$$

## IV. CONCLUSIONS

The study of quantum fields in a curved background requires choosing appropriate initial conditions, as these determine the initial state of the quantum system. There exists an appropriate adiabatic basis, in which one finds very simple expressions for the energy and pressure of the quantum field, which have a classical interpretation: the total energy of the system is the sum over modes of the number of particles of momentum $k$ times the frequency of that mode. The pressure is obtained from the conservation equation. These results can be generalized for other values of $\xi$, and no essential changes in the formalism here developed are needed.

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